Exercise 1 (The Direct Method I)**.** Let $\Omega \subset \mathbb{R}^d$ be a bounded smooth domain and $f \in L^\infty(\Omega)$. Show that the problem

$$\inf_{u \in W_0^{1,2}(\Omega)} \left(\frac{1}{2} \int_\Omega |\nabla u(x)|^2 dx - \int_\Omega f(x)u(x)dx \right)$$

admits a unique solution and compute the associated Euler-Lagrange equation.

Exercise 2 (The Direct Method II)**.** Let $\Omega \subset \mathbb{R}^d$ be a bounded smooth domain and $f \in L^\infty(\Omega)$. Show that the problem

$$\inf_{u \in W_0^{2,2}(\Omega)} \left(\frac{1}{2} \int_\Omega (\Delta u)^2 dx - \int_\Omega f\, u\, dx \right)$$

admits a unique solution and compute the associated Euler-Lagrange equation.

Hint: first show that for all $u \in C_c^\infty(\Omega)$, we have

$$\int_\Omega (\Delta u)^2 dx = \int_\Omega |\nabla^2 u|^2 dx.$$

Exercise 3 (Failure of the existence theorem for $p = 1$)**.** Let $I =]-1, 1[\subset \mathbb{R}$ and $F : W^{1,1}(I) \to \mathbb{R}$ be the functional such that

$$E(u) = \int_{-1}^1 (|u'(x)| + |u(x) - \mathrm{sgn}(x))\, dx = \int_{-1}^1 |u'(x)|dx + \int_{-1}^0 |u(x) + 1|dx + \int_0^1 |u(x) - 1|dx.$$

1. Let $g(x) = x$. Show that the infimum of E on

$$W_g^{1,1}(I) = W^{1,1}(I) \cap \left\{ u : u - g \in W_0^{1,1}(I) \right\}$$

   is equal to 2.

2. Show that $E(u) = 2$ implies that $u(x) = \mathrm{sgn}(x)$ and conclude after showing that $u \notin W^{1,1}(I)$.

Exercise 4 (Failure of the existence method for non-elliptic systems)**.** Let $\Omega =]0, 2\pi[^2 \subset \mathbb{R}^2$ and for all $u \in W^{1,2}(\Omega)$, define

$$E(u) = \frac{1}{2} \int_\Omega \left(\left(\frac{\partial u}{\partial t} \right)^2 - \left(\frac{\partial u}{\partial x} \right)^2 \right) dx\, dt$$

and define

$$m = \inf_{u \in W_0^{1,2}(\Omega)} E(u).$$

1. Show that $m = -\infty$.
2. Show that each critical point $u \in W^{1,2}(\Omega)$ of E satisfies in the distributional sense the equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0,$$

   which is known as the wave equation.

Exercise 5. Let $f_1 : M_2(\mathbb{R}) \to \mathbb{R}, \xi \mapsto (\det \xi)^2$ and $f_2 : M_2(\mathbb{R}) \to \mathbb{R}, \xi \mapsto |\xi|^4 + 16(\det \xi)^2$. Show that neither f_1 or f_2 is a convex function.